Continuity and Differentiability - Classwork

Back in our precalculus days, we dabbled in the concept of continuity. We reached a very informal definition of continuity: a curve is continuous if you can draw it without taking your pencil from the paper. This is a good "loose" definition but when one examines it closely, it is filled with holes. For instance, we know that the function $f(x) = x^2$ is a parabola and the parabola is continuous. But can we really draw it in its entirety without taking our pencil from the paper? Try it.

So we need a definition of continuity that is better than the one given above. We will use the following definition for continuity of a function.

A function is continuous at c if <u>all three</u> of the following holds:

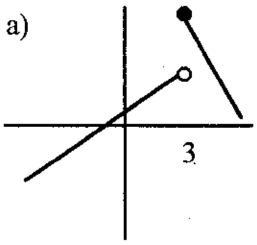
2. [redacted] $\lim_{x \to c} f(x)$ exists 1. [redacted] $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$

What we are saying is this: for the function to be continuous at some value of c:

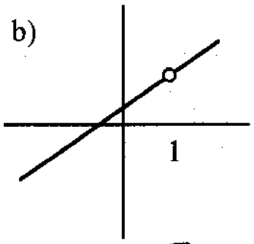
1) as x gets closer and closer to c both on the left and right, we approach the same y-value.
2) the function is defined at $x = c$.
3) the y-value limit you got in step 1 is the same as the value you get for the function at c in step 2.

If a function is continuous at all values of x then we say it is a continuous function. A function can be continuous in certain parts of its domain and discontinuous in others.

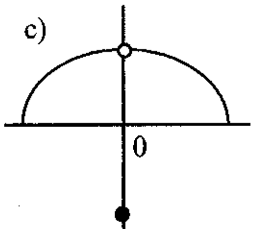
Examples: In the following graphs determine if the function $f(x)$ is continuous at the marked value of c, and if not, determine which of the 3 rules of continuity the function fails.

a)


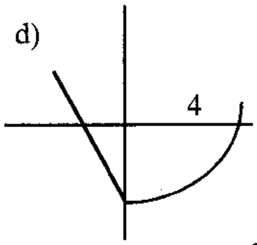
$\lim_{x \to 3} f(x)$ DNE

b)


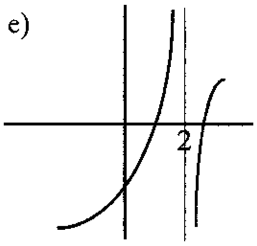
$f(1)$ DNE

c)


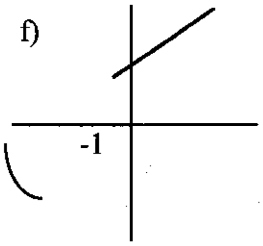
$f(0) \neq \lim_{x \to 0} f(x)$

d)


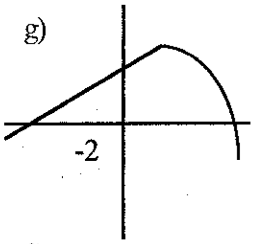
Continuous

e)


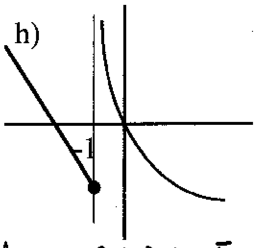
$f(2)$ DNE

f)


$f(-1)$ DNE

g)


Continuous

h)


$\lim_{x \to -1} f(x)$ DNE

Example a) above is an example of a "step" discontinuity. Notice how the function make a step at $x = 3$. Example b) is an example of a removable discontinuity (we usually call it a hole). We will see why it is called removable when we examine it in algebraic form. hole = removable discontinuity

VA and Jumps = Nonremovable discontinuity

When we examine functions in algebraic form, we can make the following conclusions:

a) all polynomials 2. $\lim\limits_{x\to c} f(x)$ exists 1. $f(c)$ exists 3. $\lim\limits_{x\to c} f(x) = f(c)$ are continuous at all values of x.

b) fractions in the form of $y = \dfrac{f(x)}{g(x)}$ are discontinuous wherever $g(x) = 0$.

c) radicals in the form of $y = \sqrt[\text{odd root}]{f(x)}$ are continuous everywhere.

d) radicals in the form of $y = \sqrt[\text{even root}]{f(x)}$ are discontinuous where $f(x) < 0$.

Examples: Find any points of discontinuity of the following functions.

a) $f(x) = -3x^2 - 5x + 1$

Continuous everywhere

b) $f(x) = \dfrac{x-2}{x^2-4}$

$f(x) = \dfrac{(x-2)}{(x-2)(x+2)}$

hole at $x = 2$

VA at $x = -2$

c) $f(x) = \sqrt[3]{x^2 + 2x - 1}$

Continuous everywhere

d) $f(x) = \sqrt{x^2 - x - 6}$

$f(x) = \sqrt{(x-3)(x+2)}$

$(-)(-) \quad (-)(+) \quad (+)(+)$

$-2 \quad 3$

Discontinuous on $(-2, 3)$

In dealing with continuity of a piecewise function, we need to examine the x-value where the rule changes.

Example 3) $f(x) = \begin{cases} x^2 - 3, x \ge 1 \\ 1 - x, x < 1 \end{cases}$

1. $f(1) = -2$
2. $\lim\limits_{x\to 1^+} x^2 - 3 = -2$

$\lim\limits_{x\to 1^-} 1 - x = 0$

$\lim\limits_{x\to 1} f(x)$ DNE

$\therefore f$ is not continuous at $x = 1$

Example 4) $f(x) = \begin{cases} x^2 + 3x - 2, x \ge -2 \\ -x^2, x < -2 \end{cases}$

1. $f(-2) = -4$
2. $\lim\limits_{x\to -2^+} x^2 + 3x - 2 = -4$

$\lim\limits_{x\to 2^-} -x^2 = -4$

$\lim\limits_{x\to 2} f(x) = -4$

3. $f(-2) = \lim\limits_{x\to 2} f(x)$

$\therefore f$ is continuous at $x = -2$

Example 5) $f(x) = \begin{cases} 3^{-x} - 1, x \ge -1 \\ \dfrac{1}{x+1}, x < -1 \end{cases}$

1. $f(-1) = 3^{-(-1)} - 1 = 2$
2. $\lim\limits_{x\to -1^+} 3^{-x} - 1 = 2$

$\lim\limits_{x\to -1^-} \dfrac{1}{x+1} = \text{DNE}$

$\lim\limits_{x\to -1} f(x)$ DNE

$\therefore f$ is not continuous at $x = -1$

Example 6) $f(x) = \begin{cases} \dfrac{x-4}{x^2-16}, x \ne 4 \\ \dfrac{1}{3x-4}, x = 4 \end{cases}$

1. $f(4) = \frac{1}{8}$
2. $\lim\limits_{x\to 4} \dfrac{\cancel{(x-4)}}{\cancel{(x-4)}(x+4)} = \dfrac{1}{8}$
3. $f(4) = \lim\limits_{x\to 4} f(x)$

$\therefore f$ is continuous at $x = 4$

Find the value of the constant k that will make the function continuous. Verify by calculator.

Example 7) $f(x) = \begin{cases} 3x + 2, x \ge 1 \\ 2k - x, x < 1 \end{cases}$

1. $f(1) = 5$
2. $\lim\limits_{x\to 1^+} 3x + 2 = 5$

$\lim\limits_{x\to 1^-} 2k - x = 2k - 1$

$2k - 1 = 5$

$2k = 6$

$\boxed{k = 3}$

Example 8) $f(x) = \begin{cases} kx^2, x \ge 2 \\ kx - 6, x < 2 \end{cases}$

1. $f(2) = 4k$
2. $\lim\limits_{x\to 2^+} kx^2 = 4k$

$\lim\limits_{x\to 2^-} kx - 6 = 2k - 6$

$4k = 2k - 6$

$2k = -6$

$\boxed{k = -3}$

Example 9) $f(x) = \begin{cases} k^2 - 12x, x \ge 1 \\ kx, x < 1 \end{cases}$

1. $f(1) = k^2 - 12$
2. $\lim\limits_{x\to 1^+} k^2 - 12x = k^2 - 12$

$\lim\limits_{x\to 1^-} k(x) = k$

$k^2 - 12 = k$

$k^2 - k - 12 = 0$

$(k-4)(k+3) = 0$

$\boxed{k = 4 \text{ or } k = -3}$

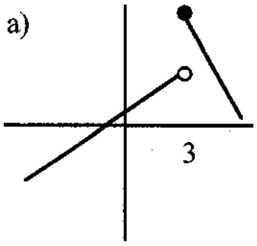
An important concept in calculus involves the concept of **differentiability**. There are several definitions that you need to know:

Definitions
Differentiability at a point: Function $f(x)$ is differentiable at $x=c$ if and only if $f'(c)$ exists. That is, $f'(c)$ is a real number.
Differentiability on an interval: Function $f(x)$ is differentiable on an interval (a,b) if and only if it is diffferentiable for every value of x on the interval (a,b).
Differentiability: Function $f(x)$ is differentiable if and only if it is differentiable at every value of x in its domain.

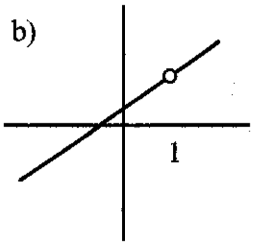
The concept of differentiability means, in laymans terms, "smooth." A differentiable curve will have no sharp points in it (cusp points) or places where the tangent line to the curve is vertical. Imagine a train traveling on a set of differentiable tracks and you will never get a derailment. Naturally, if a curve is to be differentiable, it must be defined at every point and its limit must exist everywhere. That implies the following:

Differentiability implies Continuity, Continuity does not imply Differentiability.	
If a function $f(x)$ is differentiable at $x=c$ then it must be continuous also at $x=c$.	$\boxed{D \Rightarrow C}$
However, if a function is continuous at $x=c$, it need not be differentiable at $x=c$.	$\boxed{\text{Not! } C \Rightarrow D}$
And, if a function is not continuous, then it can't be differentiable at $x=c$.	$\boxed{\text{not } C \Rightarrow \text{not } D}$

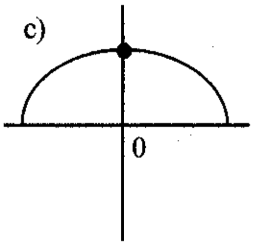
Example: determine whether the following functions are continuous, differentiable, neither, or both at the point.

a)


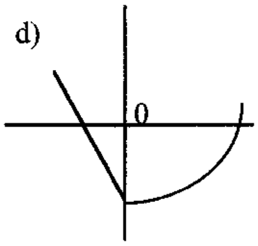
Neither

b)


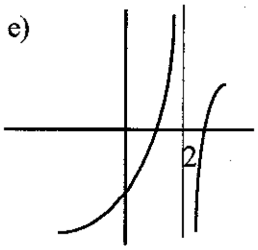
Neither

c)


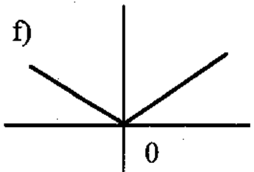
Both

d)


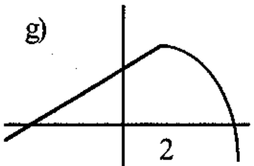
Continuous
Not Differentiable

e)


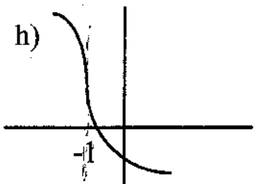
Neither

f)


Continuous
Not Differentiable

g)


Continuous
Not Differentiable

h)


Continuous
Not Differentiable

i) $f(x)=x^2-6x+1$

Both

j) $f(x)=\dfrac{x^2-x-12}{x+3}$

$f(x)=\dfrac{(x+3)(x-4)}{(x+3)}$

hole at $x=-3$
Neither

k) $f(x)=\sin x$

Both

l) $f(x)=\dfrac{\sin x}{x}$

$x \neq 0$

Neither

Example 2) Determine if $f(x)$ is continuous and/or differentiable at the value of the function where the rule changes. Sketch the function.

a) $f(x)=\begin{cases} x^2-6x+10, x\ge 2 \\ 4-x, x<2 \end{cases}$ $f'(x)=\begin{cases} 2x-6, x\ge 2 \\ -1, x<2 \end{cases}$

1. $f(2)=4-12+10=2$
2. $\lim_{x\to 2^-} 4-x=2$

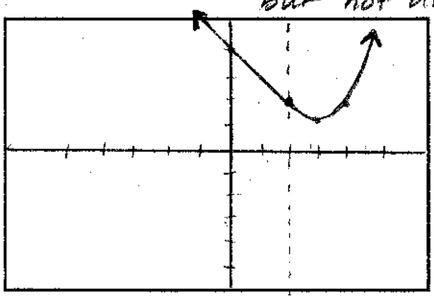
   $\lim_{x\to 2^+} x^2-6x+10=2$

   $\lim_{x\to 2} f(x)=2$
3. $f(2)=\lim_{x\to 2} f(x)$

1. $f'(2)=-2$
2. $\lim_{x\to 2^+} 2x-6=-2$

   $\lim_{x\to 2^-} -1=-1$

f is continuous but not differentiable at $x=2$



b) $f(x)=\begin{cases} x^2+x-3, x\ge -1 \\ -x-4, x<-1 \end{cases}$ $f'(x)=\begin{cases} 2x+1, x\ge -1 \\ -1, x<-1 \end{cases}$

1. $f(-1)=-3$
2. $\lim_{x\to -1^+} x^2+x-3=-3$

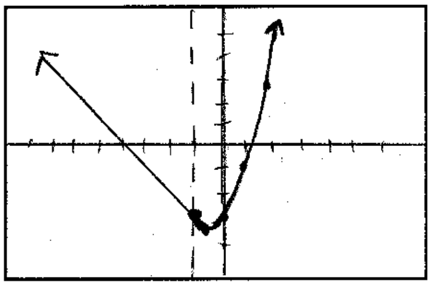
   $\lim_{x\to -1^-} -x-4=-3$

   $\lim_{x\to -1} f(x)=-3$
3. $f(-1)=\lim_{x\to -1} f(x)$

1. $f'(-1)=-1$
2. $\lim_{x\to -1^+} 2x+1=-1$

   $\lim_{x\to -1^-} -1=-1$

   $\lim_{x\to -1} f(x)=-1$

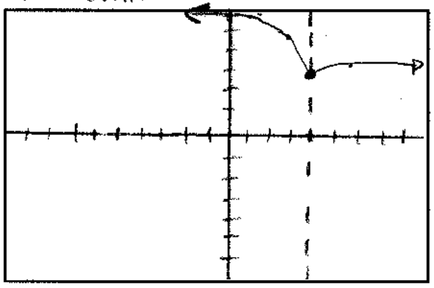


f is continuous and differentiable at $x=-1$

c) $f(x)=\begin{cases} \sqrt{x+5}, x\ge 4 \\ 4-\sqrt[3]{x-4}, x<4 \end{cases}$

1. $f(4)=3$
2. $\lim_{x\to 4^-} 4-\sqrt[3]{x-4}=4$

$\therefore$ f is not continuous nor differentiable at $x=4$



d) $f(x)=\begin{cases} \sin x, x\ge 0 \\ x-3x^2, x<0 \end{cases}$ $f'(x)=\begin{cases} \cos x, x\ge 0 \\ 1-6x, x<0 \end{cases}$

1. $f(0)=\sin 0=0$
2. $\lim_{x\to 0^+} \sin x=0$

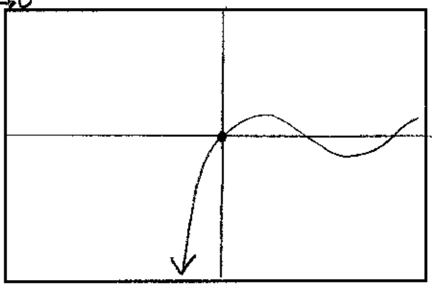
   $\lim_{x\to 0^-} x-3x^2=0$

   $\lim_{x\to 0} f(x)=0$
3. $f(0)=\lim_{x\to 0} f(x)$

1. $f'(0)=\cos 0=1$
2. $\lim_{x\to 0^+} \cos x=1$

   $\lim_{x\to 0^-} 1-6x=1$

   $\lim_{x\to 0} f(x)=1$



f is continuous and differentiable at $x=0$

Example 3) Find the values of a and b that make the function $f(x)$ differentiable.

a) $f(x)=\begin{cases} ax^2+1, x\ge 1 \\ bx-3, x<1 \end{cases}$ $f'(x)=\begin{cases} 2ax, x\ge 1 \\ b, x<1 \end{cases}$

1. $f(1)=a+1$
2. $\lim_{x\to 1^-} bx-3=b-3$

   $\lim_{x\to 1^+} ax^2+1=a+1$
3. $\lim_{x\to 1} f(x)=f(1)$

$\therefore a+1=b-3$

1. $f'(1)=2a$
2. $\lim_{x\to 1^-} b=b$

   $\lim_{x\to 1^+} 2ax=2a$
3. $f'(1)=\lim_{x\to 1} f'(x)$

   $b=2a$

thus $a+1=2a-3$

$\boxed{a=4}$ and $\boxed{b=8}$

b) $f(x)=\begin{cases} ax^3+1, x<2 \\ b(x-3)^2+10, x\ge 2 \end{cases}$ $f'(x)=\begin{cases} 3ax^2, x<2 \\ 2b(x-3), x\ge 2 \end{cases}$

1. $f(2)=b+10$
2. $\lim_{x\to 2^-} ax^3+1=8a+1$

   $\lim_{x\to 2^+} b(x-3)^2+10=b+10$
3. $f(2)=\lim_{x\to 2} f(x)$

$\therefore 8a+1=b+10$

$b=8a-9$

1. $f'(2)=12a$
2. $\lim_{x\to 2^-} 3ax^2=12a$

   $\lim_{x\to 2^+} 2b(x-3)=-2b$
3. $f'(2)=\lim_{x\to 2} f'(x)$

$\therefore 12a=-2b$

Thus $12a=-2(8a-9)$

$12a=-16a+18$

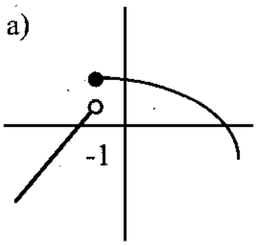
$28a=18$

$\boxed{a=\frac{9}{14}}$ and $\boxed{b=\frac{-27}{7}}$

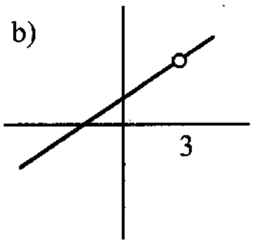
Continuity and Differentiability - Homework

1. In the following graphs determine if the function $f(x)$ is continuous at the marked value of c, and if not, determine for which of the 3 rules of continuity the function fails.

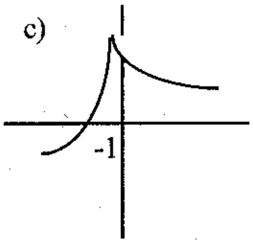
a)



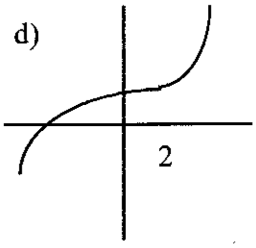
b)



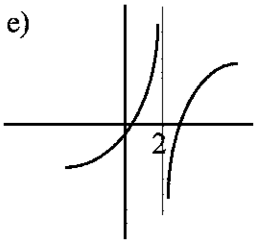
c)



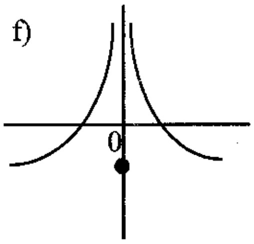
d)



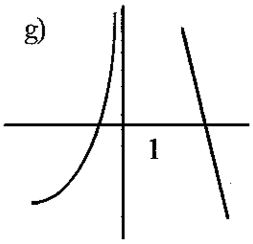
e)



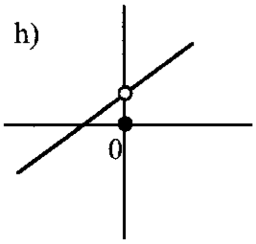
f)



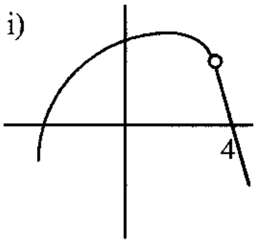
g)



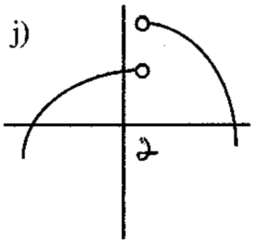
h)



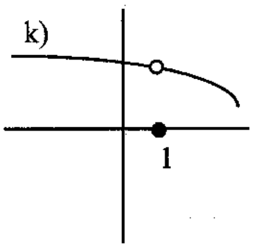
i)



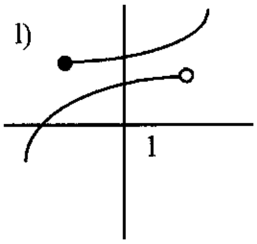
j)



k)



l)



2. Find the value of x where the function is discontinuous.

a. $f(x) = x^3 + 3^x$ b. $f(x) = \dfrac{5}{x^2 - 81}$ c. $f(x) = \dfrac{x^2 + 2x - 24}{x^2 - 36}$ d. $f(x) = \tan x$

3. Find whether the function is continuous at the value where the rule for the function changes.

a. $f(x) = \begin{cases} 8 - x^2, x < 2 \\ 6 - x, x \geq 2 \end{cases}$ b. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 1 + x, x \geq 1 \end{cases}$ c. $f(x) = \begin{cases} 2^x, x < 3 \\ 10 - x, x \geq 3 \end{cases}$

d. $f(x)=\begin{cases}2^{-x}, x<-1\\ x+3, x\geq -1\end{cases}$

e. $f(x)=\begin{cases}\dfrac{1}{x-2}, x<2\\ 3, x=2\\ x+1, x>2\end{cases}$

f. $f(x)=\begin{cases}\dfrac{x^3-x}{x^2-x}, x\neq 0,\ x\neq 1\\ 3, x=0\\ 2, x=1\end{cases}$

4. Find the value of the constant a that makes the function continuous.

a. $f(x)=\begin{cases}0.4x+2, x>1\\ 0.3x+a, x\leq 1\end{cases}$

b. $f(x)=\begin{cases}x^2, x>2\\ a-x, x\leq 2\end{cases}$

c. $f(x)=\begin{cases}9-x^2, x>2\\ ax, x\leq 2\end{cases}$

d. $f(x)=\begin{cases}ax+5, x<-1\\ ax^2, x\geq -1\end{cases}$

e. $f(x)=\begin{cases}0.4x+a^2, x<-1\\ ax+1.6, x\geq -1\end{cases}$

f. $f(x)=\begin{cases}a^2-x^2, x<2\\ 1.5ax, x\geq 2\end{cases}$

5. Let a and b stand for constants and let $f(x)=\begin{cases}b-x, x<1\\ a(x-2)^2, x\geq 1\end{cases}$

a. Find an equation relating a and b if f is to be continuous at $x = 1$.

b. Find b if a = -1. Graph and show that the function is continuous

c. Find another value of a, b where f is continuous.

6. Graph the function $f(x)=x+4+\dfrac{10^{-25}}{x-2}$ and find what appears to be the limit of $f(x)$ as x approaches 2.

7. Sketch a function having the following attributes.

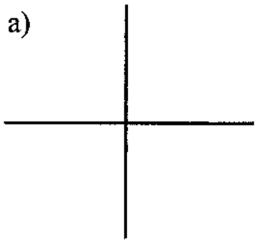
a) has a value of $f(2)$, a limit as x approaches 2, but is not continuous at $x = 2$.

b. has a step discontinuity at $x = 3$ where $f(3)=7$

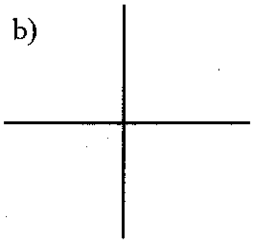
c. $\lim\limits_{x\to 4} f(x)=-2$ but the function is not continuous at $x = 4$.

d. the value of $f(-2)=3$ but there is no limit of $f(x)$ as x approaches -2 and no vertical asymptote there.

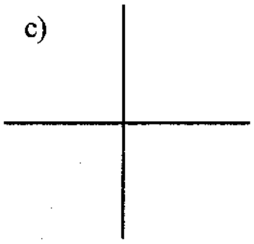
a)



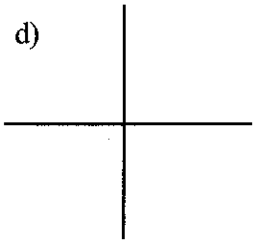
b)



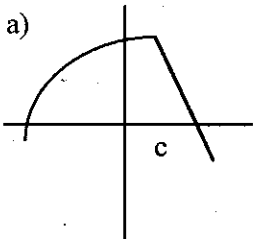
c)

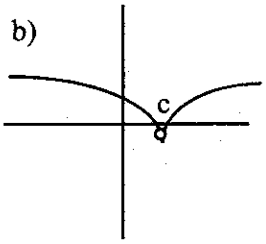


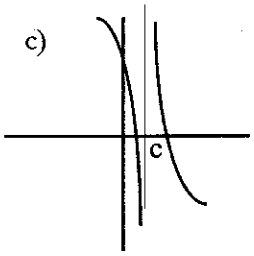
d)



8. For the following, state whether the function is continuous, differentiable, both, or neither at $x = c$

a)


b)


c)


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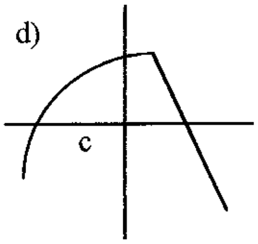
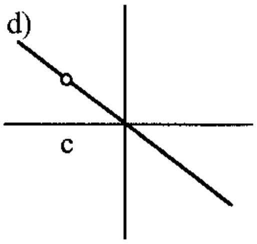
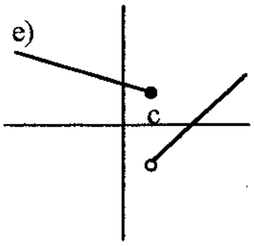
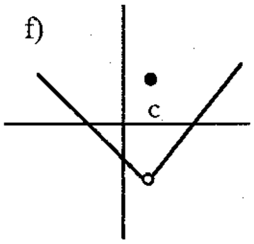
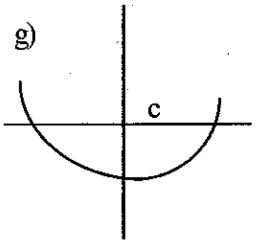


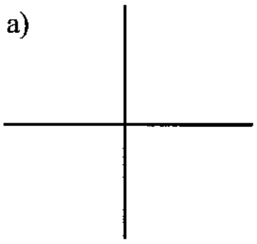
d)


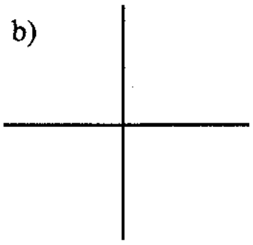
e)


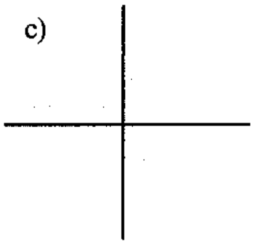
f)


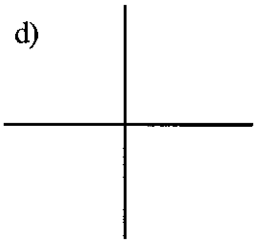
g)


9. Sketch a function having the following attributes.
   a. is differentiable and continuous at point (2, 4)
   b. is continuous at (-3, 1) but not differentiable there
   c. has a cusp at the point (-1, 3)
   d. is differentiable at (2, -4) but not continuous there.

a)


b)


c)


d)


10. For each function, $f(x)$, show work to determine whether the function is continuous or non-continuous, differentiable, or non-differentiable, and sketch the curve. Show work necessary to prove your statements.

a. $f(x) = \begin{cases} x^2, x \geq 0 \\ x, x < 0 \end{cases}$

b. $f(x) = \begin{cases} x^2 + 1, x \geq 0 \\ x^3 + 1, x < 0 \end{cases}$

c. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 2x + 2, x \geq 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

d. $f(x)=\begin{cases} x^2+x-7, x \ge 2 \\ 5x-11, x<2 \end{cases}$

e. $f(x)=\begin{cases} x^4-2x^2, x>1 \\ -1, x \le 1 \end{cases}$

f. $f(x)=\begin{cases} \sqrt{x}-3, x>1 \\ \frac{1}{2}x-\frac{5}{2}, x \le 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

g. $f(x)=\begin{cases} \sin(x), x>0 \\ x, x \le 0 \end{cases}$

h. $f(x)=\begin{cases} \cos(x), x \ge 0 \\ 1-x^2, x<0 \end{cases}$

i. $f(x)=\begin{cases} 3+(x+2)^{1/3}, x \ge -2 \\ 3-(x+2)^{2/3}, x<-2 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

11. Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a. $f(x)=\begin{cases} x^3, & x \ge 1 \\ a(x-2)^2+b, & x<1 \end{cases}$

b. $f(x)=\begin{cases} ax^2+10, & x \ge 2 \\ x^2-6x+b, & x<2 \end{cases}$

c. $f(x)=\begin{cases} a/x, & x \ge 1 \\ 12-bx^2, & x<1 \end{cases}$