# Homework 1 Solutions

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1. Consider the function $f(x) = e^x$.

(a) Derive the nth Taylor polynomial $p_n(x)$ as well as the remainder term $R_n(x)$ for the function $f(x)$, expanded about the point $x = 0$.

Let's take $x$ to be positive below for simplicity; this is okay, since we are eventually interested in $x = 1$. In terms of the Taylor polynomial $p_n(x)$, and the remainder term $R_n(x) = E_{n+1}(x)$, we have $f(x) = p_n(x) + R_n(x)$, where

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k, \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-0)^{n+1}, \quad 0 \le \xi \le x.$$

However, since $f(x) = e^x$, we have the same result for all derivatives

$$f^{(k)}(x) = e^x, \quad \forall k \ge 0 \quad \Longrightarrow f^{(k)}(0) = 1, \quad \forall k \ge 0.$$

Thus,

$$e^x = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!}e^{\xi}, \quad 0 \le \xi \le x. \tag{1}$$

(b) Using the remainder term from part (a), determine the value of $n$ needed to guarantee that $|p_n(1) - f(1)| < 10^{-5}$.

If we solve equation (1) for $R_n(x)$ evaluated at $x = 1$, we have

$$R_n(1) = f(1) - p_n(1) = \frac{e^{\xi}}{(n+1)!}, \quad 0 \le \xi \le 1$$

and would like to ensure that $|R_n(1)| < 10^{-5}$. Thus,

$$\begin{aligned} |R_n(1)| &= \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \\ &\le \max_{0 \le \xi \le 1} \left| \frac{e^{\xi}}{(n+1)!} \right| = \frac{e^1}{(n+1)!} < 10^{-5}. \end{aligned}$$

This final inequality is satisfied when $n \ge 8$. Thus, if we want the quantity $e^1 = 2.71828\ldots$ to five digits of accuracy, it suffices to approximate it with the first 8 terms of the Taylor expansion.

2. Determine which of the following sequences converges to 1 faster (clearly explain your reasoning):

$$\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \quad \text{and} \quad \lim_{x \to 0} \frac{\sin^2(x)}{x^2}$$

The first function will approach 1 more rapidly. Notice for $|x| < 1$, that $x^4 < x^2 < 1$, and so on. Therefore, if we consider $x$ near 0, we can replace $\sin(x^2)$ and $\sin^2(x)$ with their respective Taylor approximations, retaining only the first two terms, and in doing so commit only a small error. Doing so yields

$$\sin(x^2) \approx x^2 - \frac{x^6}{6},$$
$$\sin^2(x) \approx \left(x - \frac{x^3}{6}\right)^2 = x^2 - \frac{x^4}{3} + \frac{x^6}{36}.$$

We now divide these expressions by $x^2$, and bound their difference from the limit $L = 1$ to see which quantity is smaller, as $x$ becomes small. We end up with

$$\left|\frac{\sin(x^2)}{x^2} - 1\right| \leq \frac{x^4}{6}, \quad \text{whereas}$$
$$\left|\frac{\sin^2(x)}{x^2} - 1\right| \leq \frac{x^2}{3} - \frac{x^4}{36} \leq \frac{x^2}{3} \quad \text{for } x \to 0.$$

We can now say with confidence that for $|x| < \sqrt{2}$ (the interval for which the inequality $x^4/6 < x^2/3$ holds, and which contains the limit point $x = 0$), the first function will approach 1 more rapidly.

3. Compute the first 5 terms in the Taylor series (i.e., 4th order polynomial) for the following functions:

(a) $f(x) = \tanh x$, about the point $x = 0$

$$\begin{aligned}
f(x) &= \tanh(x) && \Longrightarrow && f(0) = 0 \\
f'(x) &= \text{sech}^2(x) && \Longrightarrow && f'(0) = 1 \\
f''(x) &= -2\text{sech}^2(x)\tanh(x) && \Longrightarrow && f''(0) = 0 \\
f'''(x) &= 4\text{sech}^2(x)\tanh(x) - 2\text{sech}^4(x) && \Longrightarrow && f'''(0) = -2 \\
f^{(4)}(x) &= 16\text{sech}^4(x)\tanh(x) - 8\text{sech}^2(x)\tanh^3(x) && \Longrightarrow && f^{(4)}(0) = 0
\end{aligned}$$

NOTE: the actual form of the derivatives is not unique, due to the identity $\text{sech}^2(x) + \tanh^2(x) = 1$. However, once the derivatives have been evaluated, we nonetheless end up with

$$p_4(x) = 0 + 1x + 0x^2 + \frac{-2x^3}{6} + 0x^4 = x - \frac{x^3}{6}$$

(b) $f(x) = \sin x$, about the point $x = \pi/4$

$$
\begin{aligned}
f(x) &= \sin(x) &\implies\quad & f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
f'(x) &= \cos(x) &\implies\quad & f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
f''(x) &= -\sin(x) &\implies\quad & f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
f'''(x) &= -\cos(x) &\implies\quad & f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
f^{(4)}(x) &= \sin(x) &\implies\quad & f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
\end{aligned}
$$

Once the derivatives have been evaluated, we end up with

$$p_4(x) = \frac{\sqrt{2}}{2}\left\{1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24}\left(x - \frac{\pi}{4}\right)^4\right\}$$

4. Using the results from Problem 3(b), make a single plot which contains the following:

(a) a graph of $f(x) = \sin x$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
(b) a graph of the Taylor polynomial $p_0(x)$ about $x = \pi/4$.
(c) a graph of the Taylor polynomial $p_2(x)$ about $x = \pi/4$.
(d) a graph of the Taylor polynomial $p_4(x)$ about $x = \pi/4$.

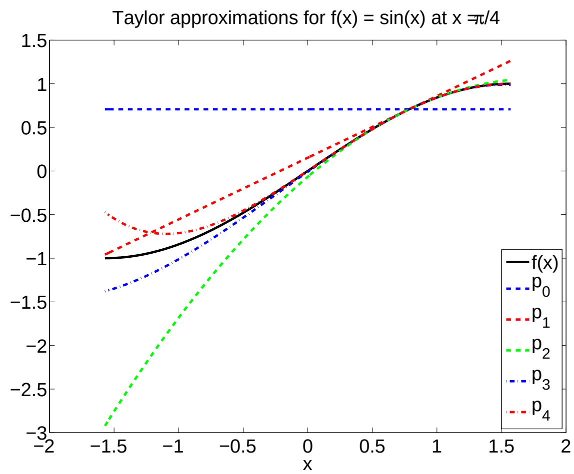
Here is a simple Matlab program to produce the desired plot:

```
c    = pi/4;                 x    = pi/2*(-1:.01:1);
z    = x-c;                  w    = ones(size(x));
f    = sin(x);               p0   =    sin(c)*w;
p1   = p0+cos(c)*z;          p2   = p1-sin(c)*z.^2/2;
p3   = p2-cos(c)*z.^3/6;     p4   = p3+sin(c)*z.^4/24;

H= figure('Position',[300 50 1300 1000]);          set(gca,'FontSize',24);
set(H,'PaperOrientation','landscape','PaperPosition',[-.43 -.2 11.75 8.5]);

plot(x,f,'k',x,p0,'--b',x,p1,'--r',x,p2,'--g',x,p3,'-.b',x,p4,'-.r','linewidth',3);
title('Taylor approxiamations for f(x) = sin(x) at x =\pi/4');
xlabel('x');             legend('f(x)','p_0','p_1','p_2','p_3','p_4',4);
saveas(H,'HW_1_3.pdf');
```

The code is also available online here; note that the online copy only contains a single command per line, making it easier to read and add comments. It is deliberately made more compact here. The plot generated is on the next page.




5. Write a program that performs Horner's algorithm. Use it to evaluate the polynomials below at the given evaluation point, and thus perform synthetic division. Interpret the output of your program in terms of the polynomial. For part (b), use your program to find all roots of $p_2(x)$ (the complete Horner's algorithm). Horner's algorithm in Matlab, displayed below. However, a fully commented code, including a description of the algorithm, all inputs and outputs, is available online here.

```
function [y,b] = Horner(a,r)
n = length(a)-1;
b=a;
for k = 2:n
    b(k) = a(k)+r*b(k-1);
end
y = a(n+1)+r*b(n);
end
```

(a) $p_1(x) = x^4 - 4x^3 + 7x^2 - 5x - 2$, at $x = 3$.

When the following commands are input:

```
>>p_1 = [1 7 4 -5 -2];
>>r_1 =3;
>>[y_1,q_1] = Horner(p_1,r_1)
```

The following output is produced:

```
y_1 = 19
q_1 = 1 -1 4 7
```

This tells us that $p_1(x) = y_1 + (x - r_1)q_1(x) = 19 + (x - 3)(x^3 - x^2 + 4x + 7)$.

(b) $p_2(x) = x^3 + 2x^2 - 5x - 6, \quad \text{at } x = 2.$

```
>>p_2 = [1 2 -5 -6];
>>r_2 =2;
>>[y_2,q_2] = Horner(p_2,r_2)
```

The following output is produced:

```
y_1 = 0
q_1 = 1 4 3
```

This tells us that $p_2(x) = y_2 + (x - r_2)q_2(x) = (x - 2)(x^2 + 4x + 3)$. The remaining two roots can be determined by hand, since $x^2 + 4x + 3 = (x + 3)(x + 1)$. Thus, we have $p_2(x) = (x - 2)(x + 1)(x + 3)$.

6. Consider the functions $f(x) = 1/(1 - x)$ and $g(t) = 1/(1 + t^2)$.

(a) Obtain the infinite Taylor series representation of $f(x)$ about the point $x = 0$.

This is a standard geometric series expansion,

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$$

which will hold for $|x| < 1$.

(b) Obtain the infinite Taylor series representation of $g(t)$ about the point $t = 0$. Do this by substituting $x = -t^2$ into the result from part (a).

To obtain the series expansion for $g(t)$, we simply make the substitution as indicated

$$g(t) = \sum_{k=0}^{\infty} (-1)^k t^{2k} = 1 - t^2 + t^4 \dots$$

This series expansion will also be defined for $|t| < 1$.

(c) Obtain the infinite Taylor series representation of $\tan^{-1}(x)$ about the point $x = 0$. Do this by integrating the resulting Taylor series in (b), since

$$\tan^{-1}(x) = \int_0^x \frac{dt}{1 + t^2}$$

Finally, we integrate the series term by term to obtain

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k + 1} = x - \frac{x^3}{3} + \dots$$

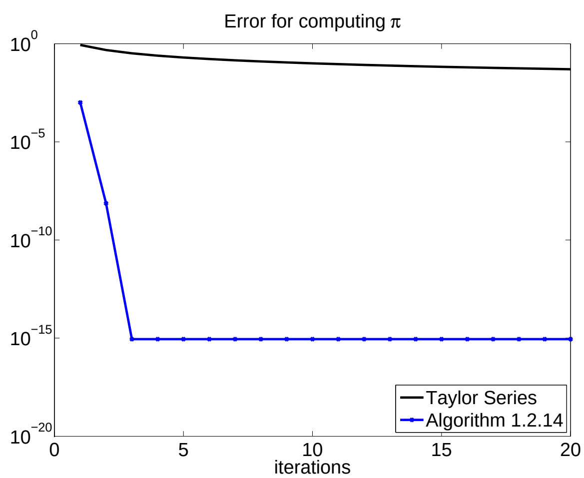
Again, the series is valid for $|x| < 1$.

7. Use the Taylor series from 6(c) to obtain an approximation to $\pi$, which is equal to $4 \tan^{-1}(1)$. Thus,

$$\pi_n \approx 4 \sum_{k=0}^{n} \frac{(-1)^k}{2k + 1}$$

Write a program that will compute approximations to $\pi$ using the formula above, and those of either problem 1.2.14, or 1.2.15. On the same set of axes, construct a semi-logarithmic plot of the absolute errors versus the number of iterations (or Taylor terms) $n$. Briefly comment on what you observe.

Below, is a plot of the error, and the code that generated it. Clearly the Taylor series produces a large error compared to that of the algorithm of problem 1.2.14. In fact, the latter of the two converges to double precision (and does not improve) within 3 or 4 iterations. The line is basically straight through the first 3 points, which means that the convergence is exponential. The Taylor series lags behind, as the error it produces after $n$ terms is approximately $2/n$ (can you show this?), indicating geometric convergence.




```
n = 1:20;        p1= 0*n;        p2= 0*n;
%algorithm 1: Taylor series
p1(1) = 4;
for k=2:20
    p1(k) =  p1(k-1)-4*(-1)^k/(2*k-1);
end
%algorithm 2: Problem 1.2.14
a    = 0;        b    = 1;        c    = 1/sqrt(2);    d    = 0.25;     e    = 1;
for k=1:20
    a    = b;        b    = (b + c )/2;    c    = sqrt(c*a);    d    = d-e*(b-a)^2;
    e    = 2*e;      f(k)=b^2/d;         p2(k)=(b+c)^2/(4*d);
end
H= figure('Position',[300 50 1300 1000]);       set(gca,'FontSize',24);
set(H,'PaperOrientation','landscape','PaperPosition',[-.43 -.2 11.75 8.5]);
semilogy(n,abs(p1-pi),'k',n,abs(p2-pi),'-xb','linewidth',3);
title('Error for computing \pi');        xlabel('iterations');
legend('Taylor Series','Algorithm 1.2.14',4);       saveas(H,'HW1_7.pdf');
```